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LETTER TO THE EDITOR

Universal scaling function for domain growth in the Glauber–Ising chain

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Abstract. The equal-time correlation function is calculated at $T=0$ for the one-dimensional Ising model with Glauber dynamics. Random initial conditions, appropriate to a sudden quench from non-zero temperature, are imposed. Averaging over initial conditions yields the scaling form $C(r,t)=f(r^2/t)$. The scaling function is given by $f(x)=\int_0^1 (\mathrm{d}y/\pi)[y(1-y)]^{-1/2}\exp(-x/4y)$, and is universal, i.e. independent of the probability distribution for the initial conditions.

The problem of domain growth, following a sudden quench of a system from an initial high-temperature state to a temperature deep in the ordered phase, is of considerable interest. Despite substantial effort [1], quantitative understanding of the evolving domain structure is limited. Much of the interest lies in the fact that this non-equilibrium phenomenon seems to exhibit many of the scaling properties familiar from the field of critical phenomena, together with a degree of universality.

Of particular interest is the equal-time two-point correlation function or, equivalently, its Fourier transform, the time-dependent structure factor $C_k(t)$. In the late stages of domain growth there is much evidence, both numerical [2] and experimental [3], for the scaling form $C_k(t)=L(t)^d g(kL(t))$, where $L(t)\sim t^n$ is the (single) length scale characterising the domain structure at time t, and d is the spatial dimensionality. While the value of the growth exponent n is relatively well understood, both for conserved [4] ($n=\frac{1}{3}$) and non-conserved [5] ($n=\frac{1}{2}$) order parameters, a first-principles calculation of the scaling function $g(x)$, for either case, is still lacking. For the non-conserved case, however, the approximate scaling function derived by Ohta *et al* [6] by consideration of domain-wall motion, fits the data reasonably well.

In view of the absence of exact results in this field, it would seem useful to consider a soluble model. This letter concerns such a soluble model of domain growth—the one-dimensional Ising model with Glauber dynamics. The Glauber dynamics imply that the order parameter is not conserved, i.e. this is a 'model A' system in the sense of Hohenberg and Halperin [7]. Since the correlation length is finite at any non-zero temperature T, the interest in this system is mainly limited to $T=0$, where equilibrium cannot be achieved in finite time: domain growth proceeds indefinitely, yielding universal non-equilibrium behaviour at long times. We have in mind, for example, a sudden quench from some $T_i>0$ to $T=0$. The up-down spin symmetry of the Hamiltonian is unbroken (on the large scale) immediately following the quench, and remains unbroken at all subsequent times for an infinite system. Instead, a universal domain structure develops, with the characteristic domain size growing as $L(t)\sim t^{1/2}$. The structure is universal in the sense that it is independent of the nature of the random

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initial conditions, provided $L(t)$ is large compared to any length scale characterising the initial conditions, e.g. the correlation length at T_i.

Although many of the results presented here can be derived as special cases of the general results given by Glauber [8], we derive them from first principles here, both to make this letter self-contained and to bring out explicitly the way in which universal scaling develops for late-stage growth at $T=0$. The key role of $T=0$, especially, is emphasised in the present work.

The model is defined by the Ising Hamiltonian

$$H=-J\sum_{i=1}^{N} S_iS_{i+1}$$

and the Glauber equation [8] for the spin configuration probability weight:

$$\begin{aligned}(\mathrm{d}/\mathrm{d}t)P(S_1,\ldots,S_N,t)\\ &= -P(S_1,\ldots,S_N,t)\sum_i\left(\frac{1-S_i\tanh\beta h_i}{2}\right)\\ &\quad+\sum_i P(S_1,\ldots,-S_i,\ldots,S_N,t)\left(\frac{1+S_i\tanh\beta h_i}{2}\right).\end{aligned}\qquad (1)$$

Here $\beta=1/T$, $h_i=J(S_{i-1}+S_{i+1})$ is the local field at site i, and we have adopted for convenience periodic boundary conditions, $S_{i+N}\equiv S_i$.

The equation of motion for the equal-time spin-spin correlation function

$$C_{i,j}(t)=\langle S_i(t)S_j(t)\rangle=\mathrm{Tr}\{P(S_1,\ldots,S_N,t)S_iS_j\}$$

is readily derived from (1):

$$(\mathrm{d}/\mathrm{d}t)C_{i,j}=-2C_{i,j}+\langle S_i\tanh\beta h_j\rangle+\langle S_j\tanh\beta h_i\rangle. \qquad (2)$$

Because the spins S_i take values ±1 only,

$$\tanh\beta h_i=\tfrac{1}{2}\tanh(2K)(S_{i-1}+S_{i+1})\to\tfrac{1}{2}(S_{i-1}+S_{i+1})$$

for $T\to0$, where $K=\beta J$. Using this result in (2) gives [8], for $T=0$ and $i\neq j$,

$$(\mathrm{d}/\mathrm{d}t)C_{i,j}=-2C_{i,j}+\tfrac{1}{2}(C_{i,j+1}+C_{i,j-1}+C_{i+1,j}+C_{i-1,j}) \qquad i\neq j. \qquad (3)$$

For $i=j$, one has simply

$$C_{i,i}=1 \qquad (4)$$

independent of t.

Solving (3) requires specifying initial conditions. These are fixed by the initial spin configuration, $C_{i,j}(0)=S_i(0)S_j(0)$. Equation (3) is greatly simplified by averaging over initial conditions (indicated by [...]), provided that the probability distribution for these is invariant under translations. Then

$$[C_{i,j}(0)]=C(|i-j|,0) \qquad (5)$$

is a function of the separation only, a property which persists to all subsequent times. Thus

$$(\mathrm{d}/\mathrm{d}t)C(r,t)=-2C(r,t)+C(r+1,t)+C(r-1,t) \qquad r\neq0 \qquad (6)$$

and

$$C(0,t)=1. \qquad (7)$$

Note that the property (5) is not too restrictive. It applies, for example, to the case where the initial state is obtained by quenching from a temperature T_i. Then [...] is equivalent to a thermal average at T_i, giving the form (5) with $C(r, 0) = \exp(-r/\xi_i)$, where ξ_i is the correlation length at T_i.

Equations (6) and (7) are readily solved by a combination of Fourier transformation in space and Laplace transformation in time. Fourier transforming first, via

$$C(r, t) = \frac{1}{N}\sum_k C_k(t)\exp(\mathrm{i}kr)$$

yields

$$\sum_k (\mathrm{d}C_k/\mathrm{d}t)\exp(\mathrm{i}kr) = -\sum_k \gamma_k C_k \exp(\mathrm{i}kr) \qquad r \neq 0 \tag{8}$$

$$\frac{1}{N}\sum_k C_k(t) = 1 \tag{9}$$

where $\gamma_k = 2(1 - \cos k) \simeq k^2$ for small k.

The next step is to multiply (8) through by $\exp(-\mathrm{i}k'r)$ and sum over $r \neq 0$. Using $\Sigma_{r\neq 0}\exp\{\mathrm{i}(k-k')r\} = N\delta_{k,k'} - 1$, and $(\mathrm{d}/\mathrm{d}t)\Sigma_k C_k = 0$, yields

$$\mathrm{d}C_k/\mathrm{d}t = -\gamma_k C_k + A(t) \tag{10}$$

$$A(t) = \frac{1}{N}\sum_k \gamma_k C_k(t) \tag{11}$$

with initial condition $C_k(0) = \Sigma_r C(r, 0)\exp(-\mathrm{i}kr)$.

Equations (10) and (11) can be solved by Laplace transform methods. Introducing

$$c_k(s) = \int_0^\infty \mathrm{d}t\, \exp(-st) C_k(t) \tag{12}$$

yields

$$c_k(s) = (C_k(0) + a(s))/(s + \gamma_k) \tag{13}$$

where $a(s)$ is the Laplace transform of $A(t)$, i.e.

$$a(s) = \frac{1}{N}\sum_k \gamma_k c_k(s). \tag{14}$$

Combining (13) and (14), and using (9), yields an equation for $a(s)$:

$$1 = s\int_{-\pi}^{\pi} \frac{\mathrm{d}k}{2\pi}\, \frac{C_k(0) + a(s)}{\gamma_k + s} \tag{15}$$

where we have converted the sum over k to an integral in the usual way. The latter step is justified if no k-mode is 'macroscopically occupied', i.e. provided the initial spin configuration contains no long-range order. If the contrary is true, such special values of k have to be extracted explicitly before converting the sum to an integral. An example of this, where the initial configuration has a net magnetisation, will be discussed below. For the moment, however, we assume no long-range order in the initial state.

We now argue that, in the scaling limit ($s \to 0$, $k \to 0$ with s/k^2 arbitrary), $C_k(0)$ is negligible compared to $a(s)$, and may be dropped from the integrand in (15). This will be justified *a posteriori*. Dropping $C_k(0)$ in (15), and replacing γ_k by k^2 in the scaling limit, gives

$$a(s) = 2s^{-1/2}. \tag{16}$$

Since the important values of k in the integral are of order $s^{1/2}$, we see that $C_k(0)$ is indeed negligible compared to $a(s)$ for $s \to 0$ provided $kC_k(0)$ vanishes for $k \to 0$. The latter condition, however, is ensured by (9).

Inserting (16) into (13), and again dropping the term in $C_k(0)$, yields the scaling function in Fourier-Laplace space:

$$c_k(s) = 2s^{-1/2}(s+k^2)^{-1}. \tag{17}$$

It remains to transform back to real time using an inverse Laplace transform. Introducing first the integral representation $s^{-1/2} = \pi^{-1/2}\int_0^\infty du\, u^{-1/2}\exp(-us)$ gives

$$C_k(t) = \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{du}{\sqrt{u}}\int_{-i\infty+\delta}^{i\infty+\delta} \frac{ds}{2\pi i}\,\frac{\exp\{s(t-u)\}}{s+k^2} \tag{18}$$

where δ is a positive infinitesimal. For $t < u$, the integration contour can be closed in the right half-plane to give zero. For $t > u$, it is closed in the left half-plane, picking up the pole at $s = -k^2$. After the change of variable $u = ty$, the final result is

$$C_k(t) = 2(t/\pi)^{1/2}\int_0^1 dy\, y^{-1/2}\exp\{-k^2 t(1-y)\}. \tag{19}$$

Equation (19) is our final result for the time-dependent structure factor in the scaling limit. It has the expected scaling form $C_k(t) = t^{1/2}g(k^2t)$, i.e. it is a representation of $\delta(k)$ for $t \to \infty$, showing the build up of a Bragg peak at large times. For $k^2t \gg 1$, the integral is dominated by y near 1, and one obtains $C_k(t) \to 2(\pi t)^{-1/2}k^{-2}$, in agreement with the expected $k^{-(d+1)}$ dependence [9] for general space dimensionality d.

The scaling limit of the spin-spin correlation function is obtained from the structure factor (19) by Fourier transformation. The result is

$$C(r,t) = \int_0^1 (dy/\pi)[y(1-y)]^{-1/2}\exp(-r^2/4ty). \tag{20}$$

Again it has the expected scaling form $C(r,t) = f(r^2/t)$. In particular, $f(0) = 1$, consistent with perfect order within a domain. The leading correction is nonanalytic at $r = 0$: $C(r,t) \simeq 1 - |r|/(\pi t)^{1/2}$ for $r^2/t \ll 1$, consistent with the k^{-2} behaviour found in the structure factor at large k. The linear dependence on separation should be a feature of the small-(r^2/t) limit in any dimension. It is equivalent to the Debye-Porod law [10] for scattering from inhomogeneous media with sharp interfaces, and implies a $k^{-(d+1)}$-dependence for the structure factor for $k^2t \gg 1$. In the opposite regime, $r^2/t \gg 1$, equation (20) gives $C(r,t) \simeq (4t/\pi r^2)^{1/2}\exp(-r^2/4t)$.

The above results can be extended to the case where domains grow at small but non-zero temperature following a quench from a higher temperature. In this case the system has a finite equilibration time: domain growth stops when the characteristic domain size reaches the equilibrium correlation length. The calculation is a straightforward generalisation of the $T = 0$ calculation presented above. The final result for the average (over initial conditions) of the time-dependent structure factor is

$$C_k(t) = 2\left(\frac{t}{\pi}\right)^{1/2}\frac{1}{k^2+\kappa^2}\int_0^1 dy\, y^{-1/2}\{\kappa^2\exp(-\kappa^2 ty) + k^2\exp(-[k^2+\kappa^2]t + k^2ty)\} \tag{21}$$

which generalises (19). Here $\kappa = \xi^{-1} \simeq \exp(2K)/2$ is the inverse correlation length. Equation (21) has the scaling form $C_k(t) = t^{1/2} g(k^2 t, \kappa^2 t)$. For $\kappa^2 t \to 0$, equation (19) is recovered, while for $\kappa^2 t \to \infty$ the equilibrium result $C_k(\infty) = 2\kappa/(k^2 + \kappa^2)$, corresponding to $C(r, \infty) = \exp(-\kappa|r|)$, is obtained.

Another generalisation is the case of initial states with long-range order. For example, if there is a non-zero net magnetisation, i.e. $[S_i(0)] = m \neq 0$, we can write $C(|i-j|, 0) = m^2 + \tilde{C}(|i-j|, 0)$, giving $C_k(0) = Nm^2 \delta_{k,0} + \tilde{C}_k(0)$. Then the $k = 0$ term has to be extracted explicitly before converting sums over k to integrals. As a result, the left-hand side of (15) becomes $1 - m^2$, and $C_k(0)$ is replaced by $\tilde{C}_k(0)$ on the right-hand side. The resulting correlation function, in the scaling limit, is $C_m(r, t) = m^2 + (1 - m^2) C(r, t)$, with $C(r, t)$ given by (20). This very simple form, in which the magnetisation does not grow with time, is special to one dimension. If the initial state contains long-range order described by a k-vector $k_0 \neq 0$, it is easy to show that (20) is recovered for times t such that $k_0^2 t \gg 1$.

So far we have considered only equal-time spin correlations. It is also possible to calculate spin–spin correlations for different times, i.e. $C_{i,j}(t, t') = \langle S_i(t) S_j(t') \rangle$. Taking $t' > t$ without loss of generality, and specialising to $T = 0$, we find instead of (3)

$$(\mathrm{d}/\mathrm{d}t') C_{i,j} = -C_{i,j} + \tfrac{1}{2}(C_{i,j+1} + C_{i,j-1}).$$

Taking an average over initial conditions to restore translational invariance, and Fourier transforming, gives

$$(\mathrm{d}/\mathrm{d}t') C_k(t, t') = -(k^2/2) C_k(t, t') \tag{22}$$

where we have once more used $1 - \cos k \simeq k^2/2$ for small k. Using the previously calculated equal-time structure factor as an initial condition at $t' = t$ yields

$$C_k(t, t') = C_k(t, t) \exp\{-k^2(t' - t)/2\} \qquad t' > t. \tag{23}$$

In particular, the autocorrelation function $C(0, t, t') = [\langle S_i(t) S_i(t') \rangle]$ is given by

$$C(0, t, t') = \int_0^{2t/(t+t')} (\mathrm{d}y/\pi)[y(1-y)]^{-1/2}$$

i.e., it depends only on the ratio t/t' in the scaling limit. Note that $C(0, t, t) = 1$, as required, while for $t' \gg t$

$$C(0, t, t') \simeq (2/\pi)(2t/t')^{1/2} \qquad t' \gg t. \tag{24}$$

Equations (23) and (24) are examples of 'multi-time scaling' of the type recently discussed by Furukawa [11]. A similar power-law decay of the autocorrelation function, $C(0, t, t') \sim t'^{-(d-\rho)/z}$ for fixed t, is also encountered following a quench to the critical point [12–14], where z is the dynamic critical exponent. It has been shown [12] that the new exponent ρ characterising non-equilibrium critical dynamics is unrelated to z and the static critical exponents. It has been calculated [12, 13] in the ε and $1/n$ expansions. It is worth noting that at $T = 0$ the one-dimensional model considered here is simultaneously in the ordered phase *and* at its critical point! This novel feature is a consequence of the system being at its lower critical dimension. From (24) we deduce (using $z = 2$) that $\rho = 0$ for $d = 1$. (Alternatively, $\rho = 0$ follows from the absence of any prefactor involving a power of t' in (23)).

An approximate form for the domain-growth scaling function for dimensions $d \geqslant 2$ has been proposed by Ohta *et al* [6]. Simply putting $d = 1$ in their general expression produces a result different from that derived in this letter. However, although the growth exponent $\frac{1}{2}$ for a non-conserved order parameter is independent of dimension, the underlying physics is somewhat different in one dimension. For $d \geqslant 2$, the mechanism driving domain growth is the curvature of the domain walls. For $d = 1$ this mechanism is absent. Instead, the Glauber dynamics considered here provides residual noise at $T = 0$ which causes the domain walls to perform independent random walks. If two domain walls occupy neighbouring bonds they annihilate. This leads to a decrease of the domain-wall density with time and, eventually, to the scaling behaviour discussed above.

In summary, the scaling function for domain growth in the one-dimensional Ising model with Glauber dynamics has been calculated. It is universal, i.e. independent of the initial spin state provided the latter contains no ferromagnetic long-range order.

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